AMPLITUDE DOMAIN-FEQUENCY REGRESSION

Francisco Parra

September 4, 2024

Introduction

The time series can be seen from an aplitude-time domain or an amplitude-frequency domain. The amplitude-frecuency domain are used to analyze properties of filters used to decompose a time series into a trend, seasonal and irregular component investigating the gain function to examine the effect of a filter at a given frequency on the amplitude of a cycle for a particular time series. The ability to decompose data series into different frequencies for separate analysis and later recomposition is the first fundamental concept in the use of spectral techniques in forecasting, such as regression espectrum band, have had little development in econometric work. The low diffusion of this technique has been associated with the computing difficulties caused the need to work with complex numbers, and inverse Fourier transform in order to convert everything back into real terms. But the problems from the use of the complex Fourier transform may be circumvented by carrying out the Fourier transform of the data in real terms, pre-multiplied the time series by the orthogonal matrix Z whose elements are defined in Harvey (1978).

The spectral analysis commences with the assumption that any series can be transformed into a set of sine and cosine waves, and can be used to both identify and quantify apparently nonperiodic short and long cycle processes (first section). In Band spectrum regression (second section) , is a brief summary of the regression of the frequency domain (Engle, 1974) The application of spectral analysis to data containing both seasonal (high frequency) and non-seasonal (low frequency) components may produce adventages, since these different frequencies can be modelled separately and then may be re-combined to produce fitted values. Durbin (1967 and 1969) desing a technique for studying the general nature of the serial dependence in a satacionary time series, that can be use to statistic contraste in This type of exercises (third section). The time-varying regression, or the regression whit the vector of parameters time.varying can be understood in this context (four section).

Spectral analysis

1

Nerlove (1964) and Granger (1969) were the two foremost researchers on the application of spectral techniques to economic time series.

The use of spectral analysis requires a change of focus from an amplitude-time domain to an amplitude-frequency domain. Thus spectral analysis commences with the assumption that any series, Xt, can be transformed into a set of sine and cosine waves such as:

$$X_t = \eta + \sum_{j=1}^{N} [a_j \cos(2\pi \frac{ft}{n}) + b_j \sin(2\pi \frac{ft}{n})] \tag{1}$$

where η is the mean of the series, a_j and b_j are the amplitude, f is the frequency over a span of n observations, t is a time index ranging from 1 to N where N is the number of periods for which we have observations, the fraction (ft/n) for different values of t converts the discrete time scale of time series into a proportion of 2 and j ranges from 1 to n where n= N/2. The highest observable frequency in the series is n/N (i.e., 0.5 cycles per time interval). High frequency dynamics (large f) are akin to short cycle processes while low frequency dynamics (small f) may be likened to long cycle processes. If we let $\frac{ft}{n} = w$ then equation (1) can be re-written more compactly as:

$$X_t = \eta + \sum_{j=1}^{N} [a_j \cos(\omega_j) + b_j \sin(\omega_j)] \tag{2}$$

Spectral analysis can be used to both identify and quantify apparently non-periodic short and long cycle processes. A given series X_t may contain many cycles of different frequencies and amplitudes and such combinations of frequencies and amplitudes may yield cyclical patterns which appear non-periodic with irregular amplitude. In fact, in such a time series it is clear from equation (2) that each observation can be broken down into component parts of different length cycles which, when added together (along with an error term), comprise the observation (Wilson and Perry, 2004).

The overall effect of the Fourier analysis of N observation to a time date is to partition the variability of the series into components at frequencies $\frac{2\pi}{N}$, $\frac{4\pi}{N}$,...,π. The component at frequency $\omega_p = \frac{2\pi p}{N}$ if called the pth harmonic. For $p \neq \frac{N}{2}$, the equivalent form to write the pth harmonic are:

$$a_p cos\omega_p t + b_p sin\omega_p t = R_p cos(\omega_p t + \phi_p)$$

.

where $R_p = \sqrt{a_p + b_p}$ and $\phi_p = tan^{-1}(\frac{-b_p}{a_p})$

The plot of $I(\omega) = \frac{NR_p^2}{4\pi}$ against ω is called the periodogram of time data. Trend will produce a peak at zero frequency, while seasonal variations produces peaks at the seasonal frquency and at integer multiples of the sesaonal frequency. Then, when a periodogram has a large peak at some frequency ω then related peaks may occurr at 2ω, 3ω,....(Chaftiel, C,2004)

Band spectrum regression

2

Hannan (1963) first proposed regression analysis in the frequency domain, later examining the use of this technique in estimating distributed lag models (Hannan, 1965, 1967). Engle (1974) demonstrated that regression in the frequency domain has certain advantages over regression in the time domain. Consider the linear regression model

$$y = X\beta + u \tag{3}$$

where X is an n x k matrix of fixed observations on the independent variables, β is a k x I vector of parameters, y is an n x 1 vector of observations on the dependent variable, and u is an n x I vector of disturbance terms each with zero mean and constant variance, σ^2.

The model may be expressed in terms of frequencies by applying a finite Fourier transform to the dependent and independent variables. For Harvey (1978) there are a number of reasons for doing this. One is to permit the application of the technique known as 'band spectrum regression', in which regression is carried out in the frequency domain with certain wavelengths omitted. Another reason for interest in spectral regression is that if the disturbances in (3) are serially correlated, being generated by any stationary stochastic process, then regression in the frequency domain will yield an asymptotically efficient estimator of β.

Engle (1974) compute the full spectrum regression with he complex finite Fourier transform based on the n x n matrix W, in which element (t, s) is given by

$w_{ts} = \frac{1}{\sqrt{n}} e^{i\lambda_t s}$, $s = 0, 1, ..., n - 1$

where $\lambda_t = 2\pi \frac{t}{n}$, t=0,1,...,n-1, and $i = \sqrt{-1}$.

Pre-multiplying the observations in observations in (3) by W yields

$$\dot{y} = \dot{X}\beta + \dot{u} \tag{4}$$

where $\dot{y} = Wy$, $\dot{X} = WX$, and $\dot{u} = Wu$.

If the disturbance vector in (4) obeys the classical assumptions, viz. $E[u] = 0$ and $E[uu'] = \sigma^2 I_n$. then the transformed disturbance vector, $\dot{u}$, will have identical properties. This follows because the matrix W is unitary, i.e., $WW^T = I$, where W^T is the transpose of the complex conjugate of W. Furthermore the observations in (4) contain precisely the same amount of information as the untransformed observations in (3).

Application of OLS to (4) yields, in view of the properties of $\dot{u}$, the best linear unbiased estimator (BLUE) of β. This estimator is identical to the OLS estimator in (3), a result which follows directly on taking account of the unitary property of W. When the relationship implied by (4) is only assumed to hold for certain frequencies, band spectrum regression is appropriate, and this may be carried out by omitting the observations in (4) corresponding to the remaining frequencies. Since the variables in (4) are complex, however, Engle (1974) suggests an inverse Fourier transform in order to convert everything back into real terms (Harvey,1974).

The problems which arise from the use of the complex Fourier transform may be circumvented by carrying out the Fourier transform of the data in real

terms. In order to do this the observations in (3) are pre-multiplied by the orthogonal matrix Z whose elements are defined as follows (Harvey,1978):

$$
z_{ts} = \begin{cases}
\left(\frac{1}{n}\right)^{\frac{-1}{2}} & t = 1 \\
\left(\frac{2}{n}\right)^{\frac{1}{2}} \cos\left[\frac{\pi t(s-1)}{n}\right] & t = 2,4,6,..(n-2) \text{ or } (n-1) \\
\left(\frac{2}{n}\right)^{\frac{1}{2}} \sin\left[\frac{\pi(t-1)(s-1)}{T}\right] & t = 3,5,7,..,(n-1) \text{ or } n \\
(n)^{\frac{-1}{2}} (-1)^{s+1} & t = n \text{ if n is even }, s = 1,...n,
\end{cases}
$$

The resulting frequency domain regression model is:

$$
y^{**} = X^{**}\beta + v \tag{5}
$$

where $y^{**} = Zy, X^{**} = ZX$ and $v = Zu$.

In view of the orthogonality of Z, $E[vv'] = \sigma^2 I_n$ when $E[uu'] = \sigma^2 I_n$ and the application of OLS to (5) gives the BLUE of β.

Since all the elements of y** and X** are real, model may be treated by a standard regression package. If band spectrum regression is to be carried out, the number of rows in y** and X** is reduced accordingly, and so no problems arise from the use of an inappropriate number of degrees of freedom.

Amplitude domain-frequency regression

Consider now the linear regression model

$$
y_t = \beta_t x_t + u_t \tag{6}
$$

where x_t is an n x 1 vector of fixed observations on the independent variable, β_t is a n x 1 vector of parameters,y is an n x 1 vector of observations on the dependent variable, and u_t is an n x 1 vector de errores distribuidos con media cero y varianza constante.

Whit the assumption that any series, y_t,x_t,β_t and ut, can be transformed into a set of sine and cosine waves such as:

$$
y_t = \eta^y + \sum_{j=1}^{N}[a_j^y \cos(\omega_j) + b_j^y \sin(\omega_j)
$$

$$
x_t = \eta^x + \sum_{j=1}^{N}[a_j^x \cos(\omega_j) + b_j^x \sin(\omega_j)]
$$

Pre-multiplying (6) by Z:

$$
\dot{y} = \dot{x}\dot{\beta} + \dot{u}
$$

(7)

where $\dot{y} = Zy, \dot{x} = Zx$, $\dot{\beta} = Z\beta$ y $\dot{u} = Zu$

The system (7) can be rewritten as (see appendix):

$$\dot{y} = Zx_t I_n Z^T \dot{\beta} + Z I_n Z^T \dot{u}$$

(8)

If we call $\dot{e} = Z I_n Z^T \dot{u}$, It can be found the $\dot{\beta}$ that minimize the sum of squared errors $E_T = Z^T \dot{e}$.

Once you have found the solution to this optimization, the series would be transformed into the time domain.

Example: Regression in frequency domain into the GDP and emploiment in Canada

The function transforms the time series in amplitude-frequency domain, order the fourier coefficient by the comun frequencies in cross-spectrum, make a band spectrum regresion of the serie y_t and x_t for every set of fourier coefficients, and select the model to pass the significance bands to periodogram cumulative (Venables and Ripley,2002).

```
> library(descomponer)
> data(PIB)
> data (celec)
> rdf(celec,PIB)

$datos
        Y          X         F         res
1   12458   65.72689 12438.74    19.26350
2   12822   67.48491 12909.66   -87.65586
3   13345   69.97484 13576.63  -231.63133
4   14288   72.98793 14383.75   -95.74524
5   15309   76.26133 15260.59    48.41183
6   16207   80.29488 16341.05  -134.05185
7   17290   83.50754 17201.62    88.37559
8   17805   85.91239 17845.81   -40.80958
9   19037   88.65090 18579.37   457.62803
10  19915   91.45826 19331.38   583.62284
11  20867   94.86328 20243.48   623.52297
12  21543   98.82299 21304.16   238.83875
13  21935  102.54758 22301.86  -366.86407
14  22253  103.69194 22608.40  -355.40283
15  21757   99.98619 21615.75   141.25334
16  22409  100.00000 21619.45   789.55406
17  20636   99.38237 21454.00  -818.00190
18  20663   97.30654 20897.95  -234.95105
19  19952   96.10971 20577.36  -625.35719

$Fregresores
     1            2
X1   1 88.15634053
X2   0 -5.68444051
```

```
X3  0 -9.44842574
X4  0 -2.21612456
X5  0 -2.62417102
X6  0 -0.79654010
X7  0 -2.39713050
X8  0 -1.53918705
X9  0 -1.43696347
X10 0 -1.18967332
X11 0 -0.69982435
X12 0 -0.92147295
X13 0 -0.82056751
X14 0 -1.14883279
X15 0 -0.66396550
X16 0 -1.26963280
X17 0 -0.21300734
X18 0 -1.09411248
X19 0 -0.01302282

$Tregresores
                1         2
 [1,] 0.2294157 15.07878
 [2,] 0.2294157 15.48210
 [3,] 0.2294157 16.05333
 [4,] 0.2294157 16.74458
 [5,] 0.2294157 17.49555
 [6,] 0.2294157 18.42091
 [7,] 0.2294157 19.15794
 [8,] 0.2294157 19.70965
 [9,] 0.2294157 20.33791
[10,] 0.2294157 20.98196
[11,] 0.2294157 21.76313
[12,] 0.2294157 22.67155
[13,] 0.2294157 23.52603
[14,] 0.2294157 23.78856
[15,] 0.2294157 22.93841
[16,] 0.2294157 22.94157
[17,] 0.2294157 22.79988
[18,] 0.2294157 22.32365
[19,] 0.2294157 22.04908

$Nregresores
[1] 2

$sse
[1] 3116177
```

```
$gcv
[1] 204869.8

> gtd(rdf(celec,PIB)$datos$res)
```

Make the forecast $Y_t(h) = \beta_0 + \beta_1 X_t(h) + ...$, you need to have the expansion for $X_t(h)$ of the development

$$X_t(h) = \eta + \sum_{j=1}^{N}[a_j \cos(\omega_j) + b_j \sin(\omega_j)] \tag{7}$$

and this development using the orthogonal transformations W to have regressors in the frequency and time domain has to be done with n observations. Therefore, we have to build a new base of regressors of size n that have to be elaborated with observations X_t, being now $t = h, h+1, h+2, ..., n, n+1, n+2, ..., n+h$.

```
> mod1=rdf(celec,PIB)
> newdata=c(100)
> predictrdf(mod1,newdata)

      fit      lwr      upr
20577.36 19641.02 21513.70
```

Seasonal Decomposition by the Fourier Coefficients

The amplitude domain-frequency regression method could be use to decompose a time series into seasonal, trend and irregular components of a time serie y_t of frequency b or number of times in each unit time interval. For example, one could use a value of 7 for frequency when the data are sampled daily, and the natural time period is a week, or 4 and 12 when the data are sampled quarterly and monthly and the natural time period is a year.

If the observation are teken at equal interval of length, $\triangle t$, then the angular frequency is $\omega = frac\pi\triangle t$. The equivalent frequency expressed in cycles per unit time is $f = \frac{\omega}{2\pi} = \frac{1}{2}\triangle t$. Whit only one observation per year, $\omega = \pi$ radians per year or $f = \frac{1}{2}$ cycle per year (1 cicle per two years), variation whit a wavelength of one year has fequency $\omega = 2\pi$ radians per year or $f = 1$ cicle per year.

For example, in a monthly time serie of $N = 100$ observation, the seasonal cycles or the wavelenghth of one year has frequency $f = \frac{100}{12} = 8,33$ cycles for 100 dates. If the time serie are 8 full year, the less seasonal frequency are 1 cycle for year, or 8 cycle for 96 observation. The integer multiplies are $2\frac{N}{12}, 3\frac{N}{12}....,$ and wavelenghth low of one year has frequency are $f < \frac{N}{12}$.

We can use (8) to estimate the fourier coefficient in time serie y_t:

$$\dot{y} = ZtI_n Z^T \dot{\beta} + ZI_n Z^T \dot{u}$$

(9)

being $t = (1, 1, ...1)_N$ or $t = (1, 2, 3, ..., N)_N$.
If $t = (1, 1, 1, ...1)_N$,

$$A = ZtI_nZ^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & 0 & 1 \end{pmatrix}$$

Then

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & 0 & 0 \end{pmatrix}$$

are use in (9) to make the regression band spectrum with the first four coefficient of fourier of the serie $\dot{y}$.

The first $2\frac{N}{12} - 1$ rows the A matrix are used to estimate the fourier coefficients corresponding to cycles of low frequency, trend cycles, and rows $2\frac{N}{12}$ and $2\frac{N}{12} + 1$ are used to estimate the fourier coefficients of 1 cicle for year. The integer multiplies re the rows $6\frac{N}{12}$, $6\frac{N}{12} + 1$, $8\frac{N}{12}$...should be used to obtain the seasonal frequency.

Example:descompose by amplitude domain-frequency regression. IPI base 2009 in Cantabria

The Industrial Price Index of Cantabria is presented in the table below

The time serie by trend an seasonal is named $TDST$. TD is calculate by band spectrum regresion of the serie y_t and the temporal index t, in which regression is carried out in low amplitude- frequency. The seasonal serie ST result to take away TD to $TDST$, and the irregular serie IR result to take away $TDST$ to y_t . The temporal index t used in the exemple are the OLS regression into IPI and the trend index $t = (1, 2, 3, ...N)_N$.

```
> data(ipi)
> descomponer(ipi,12,1)$datos
```

	y	TDST	TD	ST	IR
1	90.2	93.49148	97.29581	-3.8043288	-3.29147706
2	98.8	96.76618	97.40651	-0.6403355	2.03382281
3	92.1	105.16011	97.55957	7.6005392	-13.06010720
4	102.7	100.11383	97.73672	2.3771122	2.58616508
5	107.0	105.36545	97.91825	7.4471960	1.63455301
6	98.3	102.67619	98.08444	4.5917463	-4.37619107

7	100.9	99.14371	98.21717	0.9265446	1.75628888
8	66.3	72.41965	98.30134	-25.8816898	-6.11964836
9	101.4	100.48346	98.32624	2.1572165	0.91654243
10	111.8	107.36550	98.28651	9.0789861	4.43450007
11	111.4	105.66091	98.18276	7.4781476	5.73909316
12	85.2	86.24833	98.02170	-11.7733676	-1.04832922
13	94.4	94.02740	97.81584	-3.7884330	0.37259602
14	96.2	96.94503	97.58269	-0.6376590	-0.74503016
15	106.5	104.91231	97.34356	7.5687593	1.58768510
16	101.1	99.48917	97.12200	2.3671694	1.61083240
17	103.5	104.35813	96.94209	7.4160356	-0.85812832
18	99.9	101.39913	96.82661	4.5725269	-1.49913452
19	101.4	97.71791	96.79525	0.9226651	3.68208654
20	58.6	71.08983	96.86311	-25.7732827	-12.48982901
21	99.8	99.18765	97.03947	2.1481777	0.61234851
22	112.7	106.36795	97.32701	9.0409316	6.33205472
23	103.8	105.16833	97.72154	7.4467921	-1.36833257
24	89.0	86.48826	98.21225	-11.7239851	2.51173577
25	91.2	95.00995	98.78249	-3.7725372	-3.80995442
26	97.3	98.77602	99.41100	-0.6349825	-1.47601811
27	110.2	107.61046	100.07348	7.5369794	2.58954158
28	105.7	103.10165	100.74442	2.3572266	2.59835269
29	109.9	108.78390	101.39902	7.3848751	1.11610157
30	109.1	106.56835	102.01504	4.5533075	2.53165476
31	104.3	103.49319	102.57441	0.9187856	0.80680894
32	71.9	77.39968	103.06455	-25.6648756	-5.49967709
33	107.1	105.61838	103.47924	2.1391389	1.48162425
34	108.5	112.82176	103.81888	9.0028771	-4.32175586
35	116.6	111.50579	104.09036	7.4154366	5.09420740
36	96.5	92.63167	104.30628	-11.6746027	3.86832534
37	94.1	100.72716	104.48380	-3.7566413	-6.62715880
38	102.4	104.01079	104.64309	-0.6323060	-1.61078761
39	109.4	112.31078	104.80558	7.5051995	-2.91077599
40	109.0	107.33936	104.99208	2.3472838	1.66063840
41	113.3	112.57479	105.22108	7.3537147	0.72520870
42	116.5	110.04125	105.50717	4.5340881	6.45874503
43	107.9	106.77478	105.85988	0.9149060	1.12521823
44	76.7	80.72646	106.28293	-25.5564685	-4.02646394
45	111.0	108.90415	106.77405	2.1301001	2.09585363
46	109.3	116.29003	107.32521	8.9648227	-6.99002963
47	119.5	115.30756	107.92348	7.3840810	4.19244058
48	95.1	96.92699	108.55221	-11.6252202	-1.82698928
49	109.6	105.45181	109.19255	-3.7407455	4.14819131
50	109.0	109.19553	109.82516	-0.6296296	-0.19553175
51	125.2	117.90530	110.43188	7.4734196	7.29469750
52	104.8	113.33469	110.99734	2.3373410	-8.53468571

```
53   123.7 118.83279 111.51024   7.3225543   4.86720973
54   119.7 116.47907 111.96420   4.5148687   3.22093379
55   105.4 113.26925 112.35822   0.9110265  -7.86924913
56    84.1  87.24847 112.69653 -25.4480614  -3.14846658
57   112.1 115.10896 112.98790   2.1210613  -3.00895781
58   121.6 122.17131 113.24454   8.9267682  -0.57131077
59   120.0 120.83332 113.48059   7.3527255  -0.83331896
60    98.6 102.13448 113.71031 -11.5758378  -3.53447654
61   117.6 110.22138 113.94623  -3.7248497   7.37861665
62   117.7 113.57038 114.19733  -0.6269531   4.12962237
63   129.7 121.90910 114.46747   7.4416398   7.79089518
64   111.8 117.08157 114.75418   2.3273982  -5.28157443
65   125.2 122.33939 115.04800   7.2913939   2.86060751
66   121.2 119.82801 115.33236   4.4956493   1.37198834
67   116.8 116.49127 115.58412   0.9071470   0.30873208
68    88.2  90.43504 115.77469 -25.3396543  -2.23503871
69   113.7 117.98378 115.87175   2.1120225  -4.28377603
70   129.0 124.73008 115.84137   8.8887137   4.26992094
71   121.7 122.97177 115.65040   7.3213700  -1.27177389
72    94.4 103.74264 115.26910 -11.5264553  -9.34264377
73   110.3 110.96455 114.67351  -3.7089538  -0.66455342
74   115.3 113.22345 113.84773  -0.6242766   2.07655123
75   112.9 120.19554 112.78568   7.4098599  -7.29553587
76   122.4 113.80977 111.49232   2.3174554   8.59022509
77   116.9 117.24442 109.98419   7.2602334  -0.34442458
78   111.2 112.76564 108.28921   4.4764299  -1.56563785
79   115.0 107.34901 106.44574   0.9032674   7.65098972
80    77.1  79.26977 104.50102 -25.2312472  -2.16976916
81   106.3 104.61189 102.50890   2.1029837   1.68811145
82   115.9 109.37796 100.52731   8.8506592   6.52203544
83   106.7 105.90524  98.61523   7.2900144   0.79475657
84    83.0  85.35274  96.82981 -11.4770729  -2.35273788
85    92.2  91.53037  95.22343  -3.6930580   0.66962853
86    94.3  93.21952  93.84112  -0.6216001   1.08048013
87    96.7 100.09652  92.71844   7.3780800  -3.39651790
88    87.2  94.18741  91.87990   2.3075126  -6.98741330
89    91.0  98.56716  91.33809   7.2290730  -7.56716185
90    91.0  95.55065  91.09344   4.4572105  -4.55065228
91    95.3  92.03412  91.13474   0.8993879   3.26587643
92    70.2  66.31734  91.44018 -25.1228401   3.88265582
93    98.3  94.07301  91.97906   2.0939449   4.22699051
94   106.9 101.52634  92.71374   8.8126048   5.37365804
95   103.4 100.86057  93.60191   7.2586589   2.53942747
96    86.8  83.17132  94.59901 -11.4276905   3.62867995
97    90.5  91.98328  95.66044  -3.6771622  -1.48327720
98    91.4  96.12477  96.74369  -0.6189236  -4.72476795
```

```
99  107.7 105.15641  97.81010    7.3463001   2.54359493
100 100.6 101.12380  98.82623    2.2975698  -0.52379809
101 101.9 106.96265  99.76474    7.1979126  -5.06264944
102 105.8 105.04288 100.60489    4.4379911   0.75712158
103 101.5 102.22804 101.33254    0.8955084  -0.72804413
104  75.4  76.92534 101.93977  -25.0144330  -1.52533899
105 101.4 104.50915 102.42425    2.0849062  -3.10915268
106 109.1 111.56283 102.78828    8.7745503  -2.46283178
107 115.8 110.26517 103.03786    7.2273034   5.53483418
108  98.9  91.80330 103.18160  -11.3783080   7.09670316
109  97.6  99.56851 103.22978   -3.6612663  -1.96851275
110 102.7 102.57721 103.19346   -0.6162472   0.12278761
111 113.2 110.39836 103.08384    7.3145202   2.80163645
112 104.3 105.19939 102.91176    2.2876270  -0.89938650
113 107.6 109.85412 102.68737    7.1667522  -2.25412104
114 103.5 106.83880 102.42003    4.4187717  -3.33880379
115  97.9 103.00994 102.11831    0.8916288  -5.10993901
116  86.3  76.88402 101.79005  -24.9060259   9.41597537
117 108.4 103.51838 101.44251    2.0758674   4.88162432
118 103.5 109.81895 101.08246    8.7364958  -6.31895228
119 103.5 107.91219 100.71625    7.1959478  -4.41219319
120  89.0  89.02086 100.34979  -11.3289256  -0.02086087
121  94.5  96.34308  99.98845   -3.6453705  -1.84307991
122  97.7  99.02332  99.63689   -0.6135707  -1.32331522
123 112.9 106.58152  99.29878    7.2827404   6.31847874
124  97.6 101.25429  98.97660    2.2776842  -3.65428531
125 111.6 105.80694  98.67135    7.1355917   5.79306067
126 103.8 102.78193  98.38238    4.3995523   1.01806936
127  97.3  98.99509  98.10734    0.8877493  -1.69508506
128  86.6  73.04459  97.84221  -24.7976188  13.55540629
129  94.7  99.64840  97.58158    2.0668286  -4.94840385
130 100.3 106.01739  97.31895    8.6984413  -5.71739065
131  95.4 104.21194  97.04735    7.1645923  -8.81193975
132  85.4  85.48037  96.75991  -11.2795431  -0.08036676
133  96.3  92.82113  96.45061   -3.6294747   3.47886911
134  94.5  95.50404  96.11493   -0.6108942  -1.00403907
135  98.1 103.00152  95.75056    7.2509605  -4.90151667
136 105.0  97.62554  95.35780    2.2677414   7.37445589
137 101.0 102.04441  94.93997    7.1044313  -1.04440606
138  98.8  98.88375  94.50342    4.3803329  -0.08375036
139  91.5  94.94119  94.05732    0.8838697  -3.44119438
140  80.5  68.92406  93.61328  -24.6892117  11.57593655
141  94.6  95.24231  93.18452    2.0577898  -0.64231218
142 100.6 101.44547  92.78508    8.6603868  -0.84546802
143  91.8  99.56192  92.42868    7.1332368  -7.76191538
144  82.1  80.89749  92.12765  -11.2301607   1.20250882
```

```
145   91.8   88.08756   91.89189   -3.8043288   3.71244372
146   92.6   91.08754   91.72788   -0.6403355   1.51245564
147  100.1   99.23859   91.63805    7.6005392   0.86141476
148   95.4   93.99740   91.62029    2.3771122   1.40259993
```

```
> gdescomponer(ipi,12,1,2002,1)
```

Bibliography

Chatfield, Cris (2004). "The Analysis of Time Series: An Introduction (6th edn.)", 2004. CRC Press

Engle, Robert F. (1974), "Band Spectrum Regression", International Economic Review 15,1-11.

Hannan, E.J. (1963), "Regression for Time Series", in Rosenblatt, M. (ed.), Time Series Analysis, New York, John Wiley.

Harvey, A.C. (1978), "Linear Regression in the Frequency Domain", International Economic Review, 19, 507-512.

Venables and Ripley (2002), "Modern Applied Statistics with S" (4th edition, 2002).

Wilson, P.J. and Perry, L.J. (2004). "Forecasting Australian Unemployment Rates Using Spectral Analysis" Australian Jurnal of Labour Economics, vol 7,no 4, December 2004, pp 459-480.

Appendix

The multiplication of two harmonic series of diferent frequency:

$$[a_j \cos(\omega_j) + b_j \sin(\omega_j)] x [a_i \cos(\omega_i) + b_i \sin(\omega_i)]$$

gives the following sum:

$$a_j a_i \cos(\omega_j) \cos(\ omega_i) + a_j b_i \cos(\omega_j) \sin(\omega_i)$$

$$+ a_i b_j \sin(\omega_j) \cos(\omega_i) b_i \sin(\omega_i) + b_j b_i \sin(\omega_j) \sin(\omega_i)$$

that using the identity of the products of sines and cosines gives the following results:

$$\frac{a_j a_i + b_j b_i}{2} \cos(\omega_j - \omega_i) + \frac{b_j a_i - b_j a_i}{2} \sin(\omega_j - \omega_i)$$

$$+ \frac{a_j a_i - b_j b_i}{2} \cos(\omega_j + \omega_i) + + \frac{b_j a_i + b_j a_i}{2} \sin(\omega_j + \omega_i)$$

The circularity of ω determines that the product of two harmonics series resulting in a new series in which the Fourier coefficients it's a linear combination of the Fourier coefficients of the two harmonics series.

In the following two series:

$$y_t = \eta^y + a_0^y \cos(\omega_0) + b_0^y \sin(\omega_0) + a_1^y \cos(\omega_1) + b_1^y \sin(\omega_1) + a_2^y \cos(\omega_2) + b_2^y \sin(\omega_2) + a_3^y \cos(\omega_3)$$

$$x_t = \eta^x + a_0^x \cos(\omega_0) + b_0^x \sin(\omega_0) + a_1^x \cos(\omega_1) + b_1^x \sin(\omega_1) + a_2^x \cos(\omega_2) + b_2^x \sin(\omega_2) + a_3^x \cos(\omega_3)$$

given a matrix $\Theta^{\dot{x}\dot{x}}$ of size 8x8 :

$$\Theta^{\dot{x}\dot{x}} = \eta^x I_8 + \frac{1}{2} \begin{pmatrix} 0 & a_0^x & b_0^x & a_1^x & b_1^x & a_2^x & b_2^x & 2a_3^x \\ 2a_0^x & a_1^x & b_1^x & a_0^x + a_2^x & b_0^x + b_2^x & a_1^x + 2a_3^x & b_1^x & 2a_2^x \\ 2b_0^x & b_1^x & -a_1^x & -b_0^x + b_2^x & a_0^x - a_2^x & -b_1^x & a_1^x - a_3^x & -2b_2^x \\ 2a_1^x & a_0^x + a_2^x & -b_0^x + b_2^x & 2a_3^x & 0 & a_0^x + a_2^x & b_0^x - b_2^x & 2a_1^x \\ 2b_1^x & a_0^x + b_2^x & -b_0^x - a_2^x & 0 & -2a_3^x & -b_0^x + b_2^x & a_0^x - a_2^x & -2b_1^x \\ 2a_2^x & a_1^x + 2a_3^x & -b_1^x & a_0^x + a_2^x & -b_0^x - b_2^x & a_1^x & -b_1^x & 2a_0^x \\ 2b_2^x & b_1^x & a_1^x - 2a_3^x & b_0^x - b_2^x & a_0^x - a_2^x & -b_1^x & -a_1^x & -2b_0^x \\ 2a_3^x & a_2^x & -b_2^x & a_1^x & -b_1^x & a_0^x & -b_0^x & 0 \end{pmatrix}$$

Demonstrates that:

$$\dot{z} = \Theta^{\dot{x}\dot{x}} \dot{y}$$

where $\dot{y} = Wy$, $\dot{x} = Wx$, and $\dot{z} = Wz$.

$$z_t = x_t y_t = W^T \dot{x} W^T \dot{y} = W^T W x_t W^T \dot{y} = x_t I_n W^T \dot{y}$$

$$W^T \dot{z} = x_t I_n W^T \dot{y}$$

$$\dot{z} = W^T x_t I_n W \dot{y}$$

It is true that;

$$x_t I_n = W^T \Theta^{\dot{x} \dot{x}} W$$

and

$$\Theta^{\dot{x} \dot{x}} = W^T x_t I_n W$$